Parameterized Lifting for Sparse Signal Representations Using the Gini Index

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Abstract

Sparsity is good. We like sparsity. We can make signals more sparse by transforming them. This paper proposes a novel, two-parameter method for designing a stable wavelet basis. Our goal is to determine a basis that represents a given signal as sparsely as possible. We choose the Gini index as a measure of sparsity and sparsify a signal by iteratively lifting the wavelet basis and at each step choosing the lift that maximizes the Gini index of the representation.

1 Introduction

One of the interesting properties of wavelets is that from a given mother wavelet, a new wavelet can be made via a 'lifting procedure' and thus modified to be more effective for some purpose, (e.g. efficient representation) [1]. In [2], a single parameter wavelet lifting scheme is proposed which ensures that the resulting wavelets are biorthogonal with compact support, providing the initial wavelet is biorthogonal with compact support. The method involves the choice of a polynomial s and a scalar τ, which is constrained by the choice of s. The question as to how to best determine s and τ is open. However, given an s, we can search the range of τ for the τ that produces the lift with the a desired goal in mind.

In this paper, we parameterize the space of filters, s, by extending [2] to a two parameter system, which can be used to generate all possible length 3 (and 2) lifts. This method can be generalized to arbitrary filter lengths. Using such a system, we can search the parameter space and choose the 'best' lift with some optimization criterion or constraint in mind. As an application of our method, in this paper we determine the lifted wavelet basis that increases the sparsity of a signal's wavelet representation.

The paper is organized as follows: In Section 2 we outline the method in [2] and propose our two parameter extension, in Section 3 we define the Gini index, the measure of sparsity that we use. In Section 5 we show the error surface associated with the two-parameter lifting scheme for a known optimal lift which maximizes the sparsity of the input signal and discuss future work.

2 Parameterized Lifting

In this section we discuss the lifting scheme as proposed in [2]. The system consists of a standard analysis and synthesis filterbank as shown in Fig. 1. The input signal is analyzed by a high-pass filter, i.e. wavelet (g), and by a low-pass filter, i.e. a scaling function (h), and the high-pass coefficients are further analyzed and so on. The effectiveness of the system at sparsifying signals is based on how closely the wavelet resembles the signal at different scales.

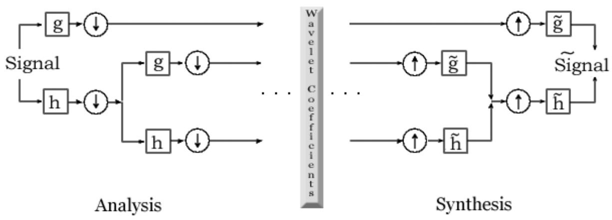

Fig. 1: Analysis and synthesis.

We can produce new analysis wavelets (g^{new}) - synthesis scaling functions ($\tilde{h}^{\text{new}}$) pairs from given pairs via lifting analysis and synthesis filters g, h, g^{new} and h^{new} respectively. In [2] the lifting scheme is defined as,

$$\begin{aligned} \hat{g}^{\text{new}}(\zeta) &= \hat{g}(\zeta) - \tau\hat{h}(\zeta)\hat{s}(2\zeta) \\ \hat{\tilde{h}}^{\text{new}}(\zeta) &= \hat{\tilde{h}}(\zeta) + \tau\hat{\tilde{g}}(\zeta)\overline{\hat{s}(2\zeta)} \end{aligned} \quad (1)$$

where $(\hat{\ })$ denotes the 2π periodic function generated from the $g, \tilde{g}, h$ and $\tilde{h}$ filters as,

$$\hat{h}(\zeta) = \sum_n h_n e^{-in\zeta} \quad (2)$$

The coefficients and length of the filter s and the value of the scalar τ are free parameters to be chosen. The lifting step that lifts the analysis wavelet g and the synthesis scaling function $\tilde{h}$ is referred to as the *primal lifting step*, henceforth referred to as simply the lifting step.

The choice of the polynomial s is arbitrary provided that $\hat{s}(0) = 0$. To begin with, we confine s to length-3, i.e. $s = [s_{-1}, s_0, s_1]$.

With the constraint that $\hat{s}(0) = 0$ and desiring symmetry to conserve linear phase i.e. $s_{-1} = s_1$, we have

$$s_1 = \tau_1[-1, 2, -1] \quad (3)$$

and to form a basis we also have

$$s_2 = \tau_2[-1, 0, 1] \quad (4)$$

The third element in the basis is $s = [1, 1, 1]$ but this does not satisfy $\hat{s}(0) = 0$.

Now we can form any length 3 s from a linear combination of (3) and (4) as below:

$$s = s_1 + s_2 = \tau_1[-1, 2, 1] + \tau_2[-1, 0, 1] \quad (5)$$

We should also note that the s of length 2 are contained in this space,

$$[0, 1, -1] = \frac{[-1, 0, 1] + [-1, 2, -1]}{2}. \quad (6)$$

We now have shifted the choice of s and τ into the choice of τ_1 and τ_2 with the constraint that s is of length at most 3. We could extend this to arbitrary length by adding basis vectors. For example, to extend to length 5, we would have

$$\begin{aligned} s &= \tau_1[0, -1, 2, -1, 0] + \tau_2[0, -1, 0, 1, 0] + \\ & \quad \tau_3[-1, 0, 0, 0, 1] + \tau_4[-1, \frac{2}{3}, , \frac{2}{3}, \frac{2}{3}, -1] \end{aligned} \quad (7)$$

Clearly any such subspace decomposition with each basis vector orthogonal to the DC vector would suffice. We prefer the proposed choices, as they have symmetric and anti-symmetric components and the simple interpretation that via adding 2 additional vectors, and by pre- and post-fixing zeros to the existing vectors, we increase the length of possible s by 2 The extension, thus, any odd length s is clear.

2.1 Limits of Stability

In [2], after choosing an initial wavelet and s, the limits on the scale of s, (i.e. the limits on τ), can be determined to ensure that the results of the lifting scheme are stable wavelets. By letting

$$s_{\theta,\tau} = \tau \cos\theta[-1, 2, -1] + \tau \sin\theta[-1, 0, 1] \quad (8)$$

and allowing θ to vary randomly we can calculate the limits on τ and empirically determine the boundary in the τ_1, τ_2 space. For example, using the Haar wavelet and scaling filter

$$\begin{aligned} g &= [0, 1, -1] \\ h &= [0, 1, 1] \end{aligned} \quad (9)$$

as the initial filter and with $\theta = 0$, we get $s_{\theta,\tau} = [-1, 2, -1]$, the limits on τ are calculated to be $-\frac{1}{2}$ and $\frac{1}{4}$, meaning that we have a stable lift for $-\frac{1}{2} < \tau < \frac{1}{4}$, i.e. the horizontal line through the origin in Fig. 2. The limits are when the associated Lawton matrix $M(\tau_1, \tau_2)$ of the lifted version of the initial filter, $\tilde{h}$ has a multiple eigenvalue at 1. There is always an eigenvalue at 1 but the system becomes unstable when the second eigenvalue reaches one. The system hence remains stable for a line segment around zero. This can be mathematically determined by solving $|I - M(\tau_1, \tau_2)| = 0$. Fig. 2 was generated by randomly varying θ and calculating the limits to generate the boundary. It should be noted that Fig. 2 is convex.

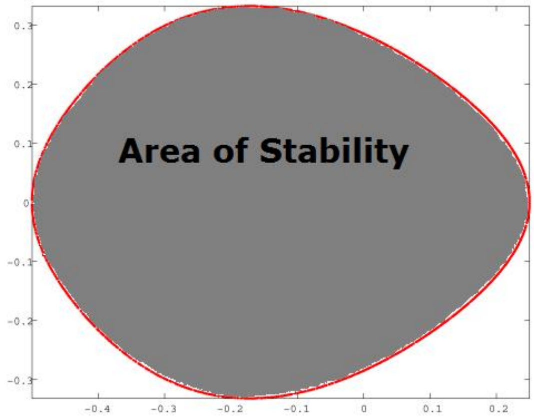

Fig. 2: The boundaries of stability in τ_1 and τ_2 space. (10,000 points)

2.2 Dual Lifting

In the lifting scheme so far we have updated the $\hat{g}$ and the corresponding $\hat{\tilde{h}}$, (1), leaving the $\hat{h}$ and $\hat{\tilde{g}}$ unchanged. It is also possible to update $\hat{h}$ and $\hat{\tilde{g}}$ with a dual-lift step,

$$\begin{aligned} \hat{\tilde{g}}^{\text{new}}(\zeta) &= \hat{\tilde{g}}(\zeta) - \tau\hat{\tilde{h}}(\zeta)\hat{s}(2\zeta) \\ \hat{h}^{\text{new}}(\zeta) &= \hat{h}(\zeta) + \tau\hat{h}(\zeta)\overline{\hat{s}(2\zeta)}. \end{aligned} \quad (10)$$

The primal lift (1) and dual lift (10) do not, in general, commute, and thus the order in which they are performed is important. After the primal or dual lift, the updated filters are used in the subsequent step. In the experiments section, when lifting will will use the following naming convention:

- *primal path*: primal (1) followed by dual (10)
- *dual path*: dual (10) followed by primal (1)

3 The Gini Index

There are several 'measures' of sparsity. Our favorite is the Gini index as we feel it captures several desirable characteristics that a sparsity measure should have. These characteristics are described here with regard to inequity of wealth distribution as that was the original application of the Gini index [3; 4].

- (Dalton's 1st Law) Robin Hood decreases sparsity. Stealing from the rich and giving to the poor, decreases the inequity of wealth distribution (assuming you don't make the rich poor and the poor rich).
- (Dalton's modified 2nd Law) Sparsity is scale invariant. Multiplying wealth by a constant factor does not alter the effective wealth distribution.
- (Dalton's 3rd Law) Adding a constant decreases sparsity. Give everyone a trillion dollars and the small differences in overall wealth are then negligible.

- (Dalton's 4th Law) Sparsity is invariant under cloning. If you have a twin population with identical wealth distribution, the sparsity of wealth in one population is the same for the combination of the two.

- Bill Gates increases sparsity. As one individual becomes infinitely wealthy, the wealth distribution becomes as sparse as possible.

- Babies increase sparsity. Adding individuals with zero wealth to a population increases the sparseness of the distribution of wealth.

With these in mind, we define the Gini index. Given coefficient data, $\mathbf{x} = \{x_1, x_2, \ldots x_N\}$, we order from smallest to largest, $|x_{(1)}| \leq |x_{(2)}| \leq \cdots \leq |x_{(N)}|$ where $(1), (2), \ldots, (N)$ are the indices of the sorting operation. The Lorenz curve is used to measure wealth distribution in society and was originally defined in [5]. We parameterize this curve with parameter p and introduce here the parameterized-Lorenz curve L_p which is the function with support $(0, 1)$, that is piecewise linear with $N+1$ points defined,

$$L_p\left(\frac{i}{N}\right) = \sum_{j=1}^{i} \frac{|x_{(i)}|^p}{\sum_{k=1}^{N} |x_k|^p}, \quad \text{for } i = 0, \ldots, N. \tag{11}$$

Note, $L_p(0) = 0$ and $L_p(1) = 1$. With $p = 2$, each point on the Lorenz curve $(x = a_0, y = b_0)$ has the interpretation that $100 \times a_0$ percent of the sorted signal coefficients captures $100 \times b_0$ percent of the total signal power. Thus, the slower the curve rises to 1, the fewer coefficients are needed to accurately represent the signal. If all coefficients were equal, which we could argue is the least sparse scenario, the curve would rise at a 45 degree angle. Thus, the area between the Lorenz curve and the 45 degree line will increase as the sparsity of the signal increases. Indeed, twice the area of this region was originally proposed (in English) in 1921 in [6] as a measure of the inequality of wealth distribution; 'Inequity in distribution' is another way of describing sparsity. The area beneath the Lorenz curve is,

$$A(\mathbf{x}) = \frac{1}{2N} \sum_{n=1}^{N} \left(L\left(\frac{n-1}{N}\right) + L\left(\frac{n}{N}\right) \right) \tag{12}$$

and twice the area between the Lorenz curve and the 45 degree, which is known as the Gini index, is then simply,

$$G_p(\mathbf{x}) = 1 - 2A(\mathbf{x}). \tag{13}$$

Some example Lorenz curves are shown in Figure 3.

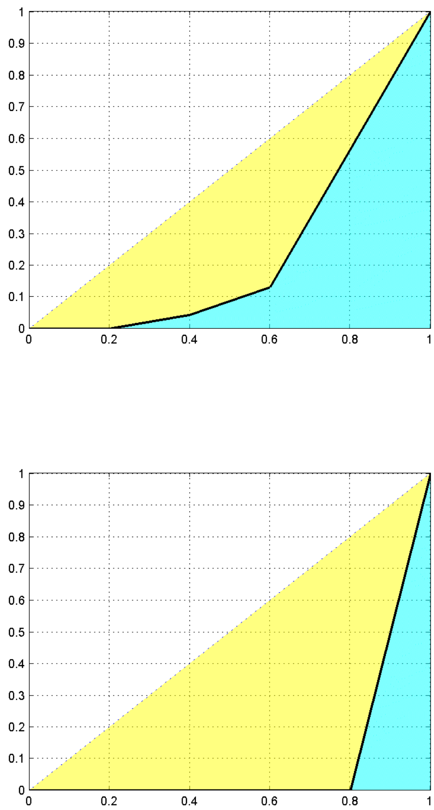

Fig. 3: Lorenz curves for [0 0 0 0 1] (upper) and [0 1 2 10 10] (lower).

4 Making signals more sparse

By using the Gini index as a measure of sparsity, the single and dual lifting steps can be used to find a wavelet basis that best represents the signal, i.e. makes the signal most sparse.

4.1 primal path two-step

The *primal path two-step* procedure for sparsifying the representation of a given signal consists of:

1. Given wavelet filters $h, g, \tilde{h}, \tilde{g}$, using (5), find the primal lift parameters ($\tau_1 = \alpha_1, \tau_2 = \beta_1$) such that a primal lift (1) with $s = \alpha_1 s_1 + \beta_1 s_2$ and $\tau = 1$ is stable and maximizes the Gini index of the resulting signal wavelet decomposition.

2. Given wavelet filters $h, g^{\text{new}}, \tilde{h}^{\text{new}}, \tilde{g}$, using (5), find the dual lift parameters ($\tau_1 = \gamma_1, \tau_2 = \delta_1$) such that a dual lift (1) with $s = \gamma_1 s_1 + \delta_1 s_2$ and $\tau = 1$ is stable and maximizes the Gini index of the resulting signal wavelet decomposition.

4.2 dual path two-step

Alternatively, we could swap the order of the primal and dual lifts, which we refer to as the *dual path two-step* procedure,

1. Given wavelet filters $h, g, \tilde{h}, \tilde{g}$, using (5), find the dual lift parameters ($\tau_1 = \alpha_2, \tau_2 = \beta_2$) such that a dual lift (1) with $s = \alpha_2 s_1 + \beta_2 s_2$ and $\tau = 1$ is stable and maximizes the Gini index of the resulting signal wavelet decomposition.

2. Given wavelet filters $h^{\text{new}}, g, \tilde{h}, \tilde{g}^{\text{new}}$, using (5), find the primal lift parameters ($\tau_1 = \gamma_2, \tau_2 = \delta_2$) such that a primal lift (1) with $s = \gamma_2 s_1 + \delta_2 s_2$ and $\tau = 1$ is stable and maximizes the Gini index of the resulting signal wavelet decomposition.

The primal or dual two-step paths can be repeated until a desired filter length is exceeded or a sparsity criteria is achieved. In this paper, we use Nelder-Mead multidimensional unconstrained nonlinear minimization (`fminsearch` in MATLAB) for the searching algorithm and set the initial lifting parameters to $(0, 0)$.

As a further alternative, the optimization above can take place over all four parameters, creating two possible one-step procedures.

4.3 primal path one-step

Given wavelet filters $h, g, \tilde{h}, \tilde{g}$, find the 4-tuple of lifting parameters $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ such that a primal lift with lift parameters (α_1, β_1) followed by a dual lift using $h^{\text{new}}, g, \tilde{h}, \tilde{g}^{\text{new}}$ with lift parameters (γ_1, δ_1) produces stable wavelets and maximizes the Gini index of the resulting signal wavelet decomposition.

4.4 dual path one-step

Given wavelet filters $h, g, \tilde{h}, \tilde{g}$, find the 4-tuple of lifting parameters $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ such that a dual lift with lift parameters (α_2, β_2) followed by a primal lift using $h^{\text{new}}, g, \tilde{h}, \tilde{g}^{\text{new}}$ with lift parameters (γ_2, δ_2) produces stable wavelets and maximizes the Gini index of the resulting signal wavelet decomposition.

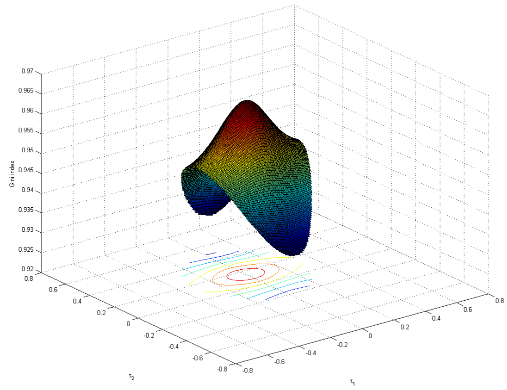

Fig. 4: Error surface (Gini index) associated with all pairings of stable (τ_1, τ_2) lifts .

5 Preliminary Result and Conclusions

As a proof-of-concept the following test was performed. First, the Haar filters were lifted using a primal lift. Then, signal synthesis was carried out by turning on one coefficient in the wavelet decomposition space (five levels of decomposition were used), thus reconstructing a synthesis basis vector. Finally, the synthesis basis vector was used as the input signal for sparsification with the system initialized to the Haar wavelet. Clearly it is of interest if the lifting method proposed here can determine the correct lift to generate the synthesis element which would be the optimally sparse representation. Figure 4 shows the (sparsity) error surface associated with all (τ_1, τ_2) primal lifts in the region of stability. The maximally sparse point corresponds to the true values used and the surface only has one maximum point.

This preliminary result is promising and future work will examine in more detail the Gini index surfaces and resulting lifting performance for more interesting signals.

References

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