L-CAMP: Extremely local high-performance wavelet representations in high spatial dimension

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***Abstract*— We study a new wavelet-based methodology for representing data on regular grids. The main attraction of this new L-CAMP methodology is in the way it scales with the spatial dimension, making it, thus, highly suitable for the representation of high dimensional data. The specific highlights of the L-CAMP methodology are three. First, it is computed and inverted by fast algorithms with linear complexity and very small constants. Second, the representation is accompanied by solid mathematical theory that reveals its performance in terms of the maximal level of smoothness that is accurately encoded by the representation. Third, the localness of the representation, measured as the sum of the volumes of the supports of the underlying mother wavelets, is extreme.**

I. Introduction

The wavelet representation is one of the major representations for data defined on regular grids. There are two main reasons for the popularity of this representation. Firstly, its discrete version is computed and inverted by a fast algorithm, the so-called Fast Wavelet Transform (FWT) [6]. Secondly, it is known to provide optimally sparse representations for the "right type" of functions/datasets (see, e.g., [3], [7]). We refer in this paper to this latter issue as "performance", and actually distinguish between two different types of performance, Jackson-type and Bernstein-type.

We are interested in this paper in wavelet representations in high-dimensions. The construction of effective wavelet representations in high spatial dimension is a challenging problem. At a first glance, the choice falls on the so-called tensor-product constructions. These constructions are readily available, their performance is well-understood, they are simple, convenient, and, to a degree, computationally effective. However, as the spatial dimension grows, such constructions become immensely non-local in space. Let us illustrate this by the following simple example. Suppose that our construction is in $\mathbb{R}^5$, and that we require Jackson-type performance $s = 2$, which essentially means that all the wavelets have two vanishing moments. A standard choice would be to use the tensor-product of Daubechies' 4-tap filters [1]. The 5-D tensor-product construction yields 31 mother wavelets. For the case here, the volume of the system, viz., the sum of the support volumes of these 31 mother wavelets is approximately 7500. This means that every point in space is visited approximately 7500 times by the wavelets within a single scale. That does not sound "local" at all.

In [4], we introduce an algorithm that, for a given fixed performance level s (we deal concretely with $s = 2, 3, 4$) and a spatial dimension n, yields a wavelet system, to which we refer as an L-CAMP system, generated by the mother wavelets $\Psi := \Psi(n, s) \subset L_2(\mathbb{R}^n)$ such that:

(1) The performance of the representation matches the given grade s. One can choose here to accept Jackson-type performance or to insist on Bernstein-type performance.

(2) The representation can be computed by the FWT, hence with linear complexity.

(3) The representation can be inverted by an algorithm which is different from, *and is at least as fast as*, the standard inversion of the FWT.

(4) A complete cycle of one decomposition step and its inversion is not only of linear complexity, but the constant in the $O(N)$ bounds, where N is the size of an initial data to be analysed, does not grow with the spatial dimension.

(5) The representation is *extremely local*. □

In the current paper we discuss some of the highlights of the L-CAMP methodology, and illustrate the general theory with the aid of two specific constructions. Column 7 of Table I lists the volumes of these two systems. The two volumes satisfy $\mathrm{vol}(\Psi) < 5$ (first system), and $\mathrm{vol}(\Psi) < 6$ (second system); in both cases, the volume is uniformly bounded independently of the dimension. This should be considered a dramatic improvement over the $\mathrm{vol}(\Psi) \approx 7500$ of the tensor-product construct that was detailed earlier; note also that the volume of all tensor-product constructions (including the Haar system) grows exponentially fast with the dimension.

Given $f : \mathbb{R}^n \to \mathbb{R}$, we denote

$$f_{j,k} := D^j E^k f = 2^{j\frac{n}{2}} f(2^j \cdot -k), \quad j \in \mathbb{Z},\, k \in \mathbb{Z}^n.$$

Here, $(Df)(t) = 2^{n/2} f(2t)$ and $(E^k f)(t) = f(t-k)$. We let χ be the characteristic function of the unit cube, and let $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^n$. Furthermore, we let $\mathbf{E} := \{0, 1\}^n$, the set of the vertices of the unit cube and let $\mathbf{E}' := \mathbf{E} \backslash 0$.

The outline of the paper is as follows. In Section II, we introduce the L-CAMP systems. In Section III, we present an algorithm for computing and inverting the L-CAMP representation, and examine its complexity. In Section IV, a rigorous discussion of the Jackson-type and Bernstein-type performance of wavelet systems is given together with the main results concerning the performance of L-CAMP systems. The proofs can be found in [4].

II. The L-CAMP system introduced

The L-CAMP class in this paper is based on the support function $\phi_c := \chi$ of the unit cube. The function ϕ_c is refinable with mask $\tau_c(\omega) = \prod_{l=1}^{n} \frac{1 + e^{-i\omega(l)}}{2}$. Here and later, we use

$$\omega := (\omega(1), \ldots, \omega(n)) \in \mathbb{R}^n$$

to denote the generic point in the frequency domain. Let h_c be the filter associated with $\overline{\tau}_c$ (i.e. $\widehat{h}_c = \overline{\tau}_c$). More general L-CAMP constructions, that use refinable functions ϕ_c other than χ, are available (see [4] for detailed discussion).

The total number of mother wavelets in our L-CAMP construction is 2^n. Recall that, in n-D, the minimal number of mother wavelets is

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$2^n - 1$, hence that we are slightly redundant. We index the L-CAMP mother wavelets by the set $\mathbf{E}$, i.e., ψ_ν, $\nu \in \mathbf{E}$.

The entire L-CAMP construction is based on two lowpass filters, and on nothing else. One of these filters is used to define the mother wavelet ψ_0, while the other one is used to define all the remaining mother wavelets. We refer to the first one as the *enhancement filter*, and to the latter one as the *main filter*.

We start by selecting the *mask* τ_e of the enhancement filter. It can be any trigonometric polynomial (in n-variables). Initially we require that the mask satisfies the relation

$$1 - \overline{\tau_e(2\omega)}\tau_c(\omega) = O(|\omega|^s), \quad \text{near the origin.} \tag{1}$$

We say that τ_e *is of order* s. While a high s is desired here, we require, at a minimum, that $s \geq 2$. The enhancement filter h_e is, then, the filter associated with the mask τ_e, i.e., $\tau_e = \widehat{h}_e$. We further would like the number of taps of h_e to be as small as possible. The third, and final, condition that the enhancement filter should satisfy is detailed later.

Next, we define the first mother wavelet, ψ_0, by the relation

$$\widehat{\psi}_0 := 2^{-n/2}\left(\widehat{\phi}_c(\cdot/2) - \overline{\tau}_e\widehat{\phi}_c\right). \tag{2}$$

Note that

$$\psi_0 = 2^{-n/2}\left(2^n\phi_c(2\cdot) - \sum_{k\in\mathbb{Z}^n}\overline{h_e(k)}\phi_c(\cdot + k)\right).$$

In order to define the remaining $2^n - 1$ mother wavelets, we choose a univariate mask τ, denote its univariate filter by h, and refer to it as the *main filter*. The main filter should also satisfy three conditions. The first condition is that it is supported on the *odd* integers, i.e.,

$$h(2m) = 0, \quad m \in \mathbb{Z}.$$

The second condition is that the filter will have high order of polynomial accuracy N: we say that h has *accuracy* N if

$$h * P = P, \quad \forall P \in \Pi^1_{N-1}. \tag{3}$$

Recall that the main filter is univariate, hence the accuracy test is conducted on univariate polynomials. A third condition that is required of the main filter is detailed later.

We lift the main filter to n-dimensions by aligning it along one of the coordinate axes. There are n different ways to do it, i.e., for $l = 1, \ldots, n$,

$$\tau_l(\omega) := \tau(\omega(l)).$$

The mother wavelet ψ_ν, $\nu \in \mathbf{E}'$, is defined by the relation

$$e^{i\nu\cdot\omega}\widehat{\psi}_\nu(2\omega) = 2^{-n/2}\left(1 - \overline{\tau_{\lceil\nu\rceil}}(\omega)\right)\widehat{\phi}_c(\omega). \tag{4}$$

Here $\lceil\ \rceil : \mathbf{E}' \to \{1, \ldots, n\}$ is a map that determines the orientation of τ that is assigned to the ν-mother wavelet. While the assignment $\lceil\ \rceil$ cannot be done at random, there is a great deal of flexibility in choosing it. One way for defining $\lceil\ \rceil$ goes as follows: for each $\nu \in \mathbf{E}'$,

$$\lceil\nu\rceil := \text{the position of the last 1-digit in the vector } \nu. \tag{5}$$

For example, for $n = 2$, $\lceil(0,1)\rceil = \lceil(1,1)\rceil = 2$ and $\lceil(1,0)\rceil = 1$. We extend the domain of the map $\lceil\ \rceil$ to $\mathbf{E}$ by defining $\lceil 0\rceil := 0$. Under this convention, the valuation map $\lceil\ \rceil$ of (5) satisfies

$$\lceil\nu - \mathbf{e}_{\lceil\nu\rceil}\rceil < \lceil\nu\rceil, \quad \nu \in \mathbf{E}'. \tag{6}$$

Here, $\mathbf{e}_l$ is the lth vector in the standard basis for $\mathbb{R}^n$.

Note that, if h is (λ-1)-tap, then $\operatorname{supp}\psi_\nu$ is the union of λ cubes that are aligned along the $\lceil\nu\rceil$-axis, each of which with volume 2^{-n}. Defining the *volume of* $\Psi \subset L_2(\mathbb{R}^n)$ to be

$$\operatorname{vol}(\Psi) := \sum_{\psi\in\Psi}\operatorname{vol}(\operatorname{supp}\psi),$$

we obtain that the volume of the L-CAMP mother wavelet set

$$\Psi := \{\psi_\nu : \nu \in \mathbf{E}\} \tag{7}$$

is

$$\operatorname{vol}(\Psi) = \operatorname{vol}(\operatorname{supp}\psi_0) + \frac{\lambda(2^n - 1)}{2^n}.$$

That is, the total volume of Ψ is bounded by $\operatorname{vol}(\operatorname{supp}\psi_0) + \lambda$.

We mentioned so far two conditions that we require of the enhancement filter, and of the main filter. Only one additional condition is required here, but it is not as simple as the ones above.

Performance conditions. Our performance analysis of the L-CAMP system is based on the following parameters:

- The order $s \geq 2$ of the enhancement mask (cf. (1)).
- The accuracy $N \geq 2$ of the main filter (cf. (3)).
- The Hölder smoothness α of the n-dimensional refinable function $\widetilde{\phi}$ associated with the mask

$$\tau_e\left(\prod_{l=1}^n \frac{1 + \tau_l}{2}\right). \tag{8}$$

For Jackson-type performance, we need to assume that

$$\alpha > 0. \tag{J}$$

The performance is then related to $\min\{s, N\}$. In all our concrete constructions, this minimum is s.
Bernstein-type performance is related to $\min\{s, N, \alpha\}$, which, again, will coincide in our constructions with $\min\{s, \alpha\}$. So, for this type of performance we desire that

$$\alpha \geq s, \tag{B}$$

or at least that α does not lag far behind s. □

III. Fast algorithms for computing and inverting the representation

The L-CAMP representation can be computed and inverted by fast algorithms with linear complexity and small constants.

L-CAMP decomposition and reconstruction algorithms. Let h_e be the enhancement filter and let h be the main filter. For $l = 1, \ldots, n$, let h_l be the lifting of h to an n-dimensional filter in the l-coordinate direction. Let $\lceil\ \rceil$ be the valuation map from (5). Then:

input $y_0 : \mathbb{Z}^n \to \mathbb{C}$

(1) Decomposition:

for $j = -1, -2, \ldots, j_0$
  $y_j(k) = 2^{-n}\sum_{\mu\in\mathbf{E}} y_{j+1}(2k + \mu)$, $k \in \mathbb{Z}^n$ $(\star)$
  if $k \in 2\mathbb{Z}^n$
    $d_{j+1}(k) = y_{j+1}(k) - (h_e * y_j)(k/2)$
  end
  if $k \in \nu + 2\mathbb{Z}^n$ where $\nu \in \mathbf{E}'$
    $d_{j+1}(k) = y_{j+1}(k) - (h_{\lceil\nu\rceil} * y_{j+1})(k)$
  end
end

(2) Reconstruction:

for $j = j_0, \ldots, -1$
  if $k \in 2\mathbb{Z}^n$
    $y_{j+1}(k) = d_{j+1}(k) + (h_e * y_j)(k/2)$

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    end
    for ⌈ν⌉ = 1, ···, n  (**)
        if  k ∈ ν + 2Z^n
            y_{j+1}(k) = d_{j+1}(k) + (h_{⌈ν⌉} * y_{j+1})(k)
        end
    end
end
```

We note that the resulted MRA $(y_j)_{j\le 0}$ from the line marked by $(\star)$ is the MRA associated with χ, that is, assuming $y_0(k) = \langle f, \chi_{0,k}\rangle$, $k \in \mathbb{Z}^n$, for some function f, it follows that

$$y_j(k) = 2^{jn/2}\langle f, \chi_{j,k}\rangle, \quad j < 0,\ k \in \mathbb{Z}^n.$$

After that line, the rest of the decomposition step computes the *detail coefficients* d_{j+1}. The interpretation of the detail coefficients is standard up to normalization, i.e., retaining the same assumption on the initial data y_0, one proves that for $j < 0$, $\nu \in \mathbf{E}$, $k \in \mathbb{Z}^n$,

$$d_{j+1}(\nu + 2k) = 2^{(j+1)n/2}\langle f, (\psi_\nu)_{j,k}\rangle.$$

The reconstruction step does not resemble its FWT counterpart. The crucial step in the reconstruction is the *for loop* marked by $(\star\star)$. We observe that, if $h_{\lceil\nu\rceil}(l) \neq 0$, then l must be of the form

$$l = a\,\mathbf{e}_{\lceil\nu\rceil},$$

where $\mathbf{e}_{\lceil\nu\rceil}$ is the unit vector in the $\lceil\nu\rceil$-coordinate direction, and a is an *odd* number. This means that all the values of y_{j+1} that are needed for the computation of $(h_{\lceil\nu\rceil} * y_{j+1})(k)$, $k \in \nu + 2\mathbb{Z}^n$, lie in $\nu - \mathbf{e}_{\lceil\nu\rceil} + 2\mathbb{Z}^n$. Since $\lceil\nu - \mathbf{e}_{\lceil\nu\rceil}\rceil < \lceil\nu\rceil$ for any $\nu \in \mathbf{E}'$ from (6), we have already recovered those values of y_{j+1} previously, hence we are able to compute $y_{j+1}(k)$ as above.

Complexity. We measure complexity by counting the number of "operations" needed in order to fully derive y_j and d_{j+1} from y_{j+1}, and add the number of operations needed for the inversion. Here, we define "an operation" as the need to fetch an entry from some of our arrays/vectors. Thus, for example, computing one entry in y_j from y_{j+1} as in $(\star)$ requires 2^n operations.

Obviously, the complexity here is linear, i.e., $\sim CM$, with M the number of non-zero entries in y_0, and C some constant. Our goal is to estimate that constant: since M is expected to grow exponentially fast with the dimension, we need, at least, to control very tightly that constant! So, we actually compute the mean number of operations per one single entry in y_0.

We observe that the number of operations required to process the portion of y_j that lies in a cube of lengthsize 2 is about

$$2^n + 2(1 + \text{ tap-size of } h_e) + 2\lambda(2^n - 1).$$

This means that the cost per entry of performing one complete cycle of decomposition/inversion is bounded by

$$1 + 2(\lambda + 2^{-n}(1 + \text{ tap-size of } h_e)).$$

□

IV. Performance analysis

A. The performance of wavelet frames

Let Ψ be a finite subset of $L_2(\mathbb{R}^n)$. The *wavelet system* generated by the *mother wavelets* Ψ is the family

$$X(\Psi) := \{\psi_{j,k} :\ \psi \in \Psi,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}.$$

The *analysis operator* is defined as

$$T^*_{X(\Psi)} : f \mapsto (\langle f, x\rangle)_{x\in X(\Psi)}.$$

$X(\Psi)$ is a *Bessel system* if $T^*_{X(\Psi)} : L_2(\mathbb{R}^n) \to \ell_2(X(\Psi))$ is bounded. A Bessel system $X(\Psi)$ is a *frame* if the analysis operator is bounded above and below.

Next, we illustrate the way the "performance" of a wavelet frame $X(\Psi)$ may be graded, and use the L_2-setup to this end. For $\alpha > 0$, let $W_2^\alpha(\mathbb{R}^n)$ be the usual Sobolev space. That is, $W_2^\alpha(\mathbb{R}^n)$ is the set of functions $f \in L_2(\mathbb{R}^n)$ s.t. $|f|_{W_2^\alpha(\mathbb{R}^n)} := \|(|\cdot|^\alpha \widehat{f}\,)^\vee\|_{L_2(\mathbb{R}^n)} < \infty$. We would like first the wavelet frame $X(\Psi)$ to satisfy

$$\sum_{\psi\in\Psi} \|T^*_{X(\psi)} f\|_{\ell_2(\alpha)} \le A_\alpha |f|_{W_2^\alpha(\mathbb{R}^n)}, \quad \forall f \in W_2^\alpha(\mathbb{R}^n). \tag{9}$$

Here,

$$\|T^*_{X(\psi)} f\|^2_{\ell_2(\alpha)} := \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} 2^{2j\alpha}|\langle f, \psi_{j,k}\rangle|^2.$$

The supremum of all $\alpha > 0$ for which (9) holds is the *Jackson-type performance* s_J of the wavelet frame $X(\Psi)$. It is known that the essential condition Ψ needs to satisfy for having "performance-grade" s_J is that each $\psi \in \Psi$ has s_J *vanishing moments* :

$$\widehat{\psi} = O(|\cdot|^{s_J}), \quad \text{near the origin.}$$

Another way to measure the performance of $X(\Psi)$ is to insist that, in addition to (9), the inverse inequality holds as well:

$$\sum_{\psi\in\Psi} \|T^*_{X(\psi)} f\|_{\ell_2(\alpha)} \ge B_\alpha |f|_{W_2^\alpha(\mathbb{R}^n)}, \quad \forall f \in L_2(\mathbb{R}^n). \tag{10}$$

The supremum of all $\alpha > 0$ for which (9) and (10) hold is the *Bernstein-type performance* s_B of the wavelet frame $X(\Psi)$. The value of s_B is not connected directly to any easy-to-check property of the system $X(\Psi)$. As a matter of fact, the value of s_B is related to the smoothness of the *dual frame* $X(\Psi^d)$, which we now describe.

First, one defines a map $\Psi \ni \psi \mapsto \psi^d \in L_2(\mathbb{R}^n)$, and extends it naturally to $X(\Psi)$ (i.e., $(\psi_{j,k})^d := (\psi^d)_{j,k}$). Assume that $X(\Psi^d)$ is also a frame. The frame $X(\Psi^d)$ is then said to be *dual* to $X(\Psi)$ if one has the *perfect reconstruction property*:

$$f = T_{X(\Psi^d)} T^*_{X(\Psi)} f = \sum_{x\in X(\Psi)} \langle f, x\rangle x^d, \quad f \in L_2(\mathbb{R}^n).$$

Here, $T_{X(\Psi^d)}$ is the *synthesis operator* :

$$T_{X(\Psi^d)} : \mathbb{C}^{X(\Psi^d)} \ni a \mapsto \sum_{x\in X(\Psi^d)} a(x)x.$$

Thus, one strives to build wavelet frame systems that have a high number of vanishing moments, and have smooth dual frames. This brings us to the question of how, actually, wavelet systems are constructed. The most general recipe in this regard is known as the Oblique Extension Principle (OEP, [2]). However, in this paper, we will need its special and simpler case, the Unitary Extension Principle (UEP). Both lead to the simultaneous construction of a frame and its dual frame. We describe now the UEP. One uses the tool of MultiResolution Analysis (MRA)([6], [7], [9], [2]): one begins with a *refinable* function $\phi \in L_2(\mathbb{R}^n)$, viz., a function whose Fourier transform satisfies a relation

$$\widehat{\phi}(2\cdot) = \tau\widehat{\phi},$$

for some 2π-periodic τ known as the *refinement mask*. One subsequently defines $V_0 := V_0(\phi) \subset L_2(\mathbb{R}^n)$ to be the closed linear span of $(E^k\phi)_{k\in\mathbb{Z}^n}$, and $V_j := V_j(\phi) := D^jV_0$, $j \in \mathbb{Z}$. The mother wavelets $\Psi = \{\psi_1, \ldots, \psi_r\}$ are then (carefully) selected from V_1. This implies that every mother wavelet $\psi_i \in \Psi$ must satisfy a relation of the form

$$\widehat{\psi_i}(2\cdot) = \tau_i\widehat{\phi},$$

TABLE I

THE TWO L-CAMP SYSTEMS THAT ARE CONSTRUCTED IN THIS PAPER. COLUMN 1 ENUMERATES THOSE SYSTEMS.

L-CAMP SYSTEMS	h_e			λ	tap-size of h_e = vol(supp ψ_0)	vol(Ψ)	Hölder smoothness of $\widetilde{\phi} \geq$	s_J	s_B
	-1	0	**1**						
I		$\frac{3}{4}$	$\frac{1}{4}$	3	2	$2+3\cdot(2^n-1)/2^n$	1.4150	2	$\geq$ 1.4150
II	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	3	3	$3+3\cdot(2^n-1)/2^n$	2.4150	2	2

for some 2π-periodic τ_i. The dual mother wavelets Ψ^d are constructed similarly, using another refinable function ϕ^d with mask τ^d, and corresponding wavelet masks $(\tau_i^d)_{i=1}^r$.

Suppose now that the two systems $X(\Psi)$ and $X(\Psi^d)$ are known to be, each, a Bessel system, and they satisfy the Mixed Unitary Extension Principle (MUEP) :

$$\tau(\cdot+\gamma)\overline{\tau^d} + \sum_{i=1}^{L} \tau_i(\cdot+\gamma)\overline{\tau_i^d} = \begin{cases} 1, & \gamma = 0, \\ 0, & \gamma \in \{0,\pi\}^n \backslash 0, \end{cases} \tag{11}$$

and $\widehat{\phi}(0) = \widehat{\phi}^d(0) = 1$. Then $X(\Psi)$ and $X(\Psi^d)$ form a pair of a wavelet frame and a dual wavelet frame [8]. We refer then to the pair $(X(\Psi), X(\Psi^d))$ as a (UEP) bi-framelet.

B. The performance of L-CAMP systems

We associate here the L-CAMP system with a dual system. We first define the following partial ordering on $\mathbf{E}$:

$$\nu' \geq \nu \quad \Longleftrightarrow \quad \nu'(l) = \nu(l), \quad l = 1, \ldots, \lceil \nu \rceil,$$

where $\lceil\ \rceil$ is defined as in (5). In particular, $\nu' \geq 0$ for all $\nu' \in \mathbf{E}$. Given a vector $a \in \mathbb{C}^n$, we define

$$a^\nu := \prod_{l=1}^{n} a(l)^{\nu(l)}.$$

Lemma 1: For $\nu \in \mathbf{E}$, let t_ν be the wavelet mask that corresponds to the mother wavelet ψ_ν defined in (2) and (4). Let ξ be any trigonometric polynomial such that $\xi(0) = 1$ and define a new refinement mask

$$\tau^d := \tau_e(2\cdot)\tau_r \left(1 + \xi\left(1 - \frac{\tau_e(2\cdot)}{2^n}\sum_{\nu\in\mathbf{E}} e_\nu \boldsymbol{\tau}^\nu\right)\right), \tag{12}$$

where $\tau_r := \prod_{l=1}^n (1+\tau_l)/2$, $\boldsymbol{\tau} := (\tau_1, \tau_2, \ldots, \tau_n)$, and $e_\nu(\omega) := e^{i\nu\cdot\omega}$ for $\omega \in \mathbb{T}^n$. We also define dual wavelet masks

$$t_\nu^d = 2^{-n/2} e_{-\nu} \sum_{\nu'\geq\nu} \boldsymbol{\tau}^{\nu'-\nu}\left(1 - \xi\tau_e(2\cdot)\tau_r e_{\nu'}\right), \quad \nu \in \mathbf{E}. \tag{13}$$

Then the masks $(\tau_c, (t_\nu)_{\nu\in\mathbf{E}})$ and $(\tau^d, (t_\nu^d)_{\nu\in\mathbf{E}})$ satisfy the MUEP condition (11), i.e.

$$\tau_c(\cdot+\gamma)\overline{\tau^d} + \sum_{\nu\in\mathbf{E}} t_\nu(\cdot+\gamma)\overline{t_\nu^d} = \begin{cases} 1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma \in \{0,\pi\}^n\backslash 0. \end{cases} \qquad \square$$

We are now ready to present our performance analysis of L-CAMP systems. We approach the performance analysis as follows. We first fix an integer $s \geq 2$. We then require the enhancement mask τ_e to be of order s (i.e. to satisfy (1)), and require the main filter h to have accuracy s as well (i.e., to satisfy (3) for $N := s$). Now suppose that we construct the L-CAMP wavelet system using τ_e and h as explained in Section II. Let Ψ be the L-CAMP mother wavelet set as in (7). Then we see that each of the mother wavelets has s vanishing moments: for ψ_0 this is due to the order of τ_e, and for all the other mother wavelets this is due to the accuracy of the main filter h.

The other important information needed for the performance analysis of $X(\Psi)$ is the smoothness of the dual system. To this end, we will show that it suffices to know the smoothness of the refinable function $\widetilde{\phi}$, the refinable function associated with the mask $\tau_e\tau_r$, (cf. (8) and (12)). We note that the standard performance analysis will hinge on the smoothness of the more complicated ϕ^d associated with τ^d (see, e.g., [7], [5]); thus the reduction of the performance analysis to the smoothness of $\widetilde{\phi}$ is an important step here.

Here we use the usual Hölder smoothness to measure smoothness. For $\eta > 0$ non-integer, there is a standard interpretation of what Hölder smoothness η means. For positive integers $M = 1, 2, \cdots$, we say that f has Hölder smoothness M if f has Hölder smoothness η for all $0 < \eta < M$.

Theorem 1: Let $s \geq 2$ be an integer. Assume that we have an L-CAMP system that satisfies (1) for the given s, and (3) for $N := s$. Suppose that $\widetilde{\phi}$ has Hölder smoothness η for some $\eta > 0$. Then $X(\Psi)$ provides $s_J \geq s$ and $s_B \geq \min\{s, \eta\}$. $\square$

The proof of the above theorem invokes the following lemma, which might be of independent interest.

Lemma 2: Let $s \geq 2$ be an integer. Assume that we have an L-CAMP system that satisfies (1) for the given s, and (3) for $N := s$. Suppose that $\widetilde{\phi}$ has Hölder smoothness η for some $\eta > 0$. Then for every $0 < \alpha < \eta$, there exists a wavelet frame $X(\Psi^d)$ associated with a refinable function ϕ^d that corresponds to the mask τ^d in (12), so that the pair $(X(\Psi), X(\Psi^d))$ is a (UEP) bi-framelet and each element in Ψ^d has Hölder smoothness α. $\square$

From Theorem 1 (for any $\eta > 0$ and $\eta := s$), we see that the L-CAMP systems satisfying the assumptions in **Performance conditions** (in Section II) have performance grade at least s, in the Jackson sense if we assume (J), and in the Bernstein sense if we assume (B).

In Table I, the two L-CAMP systems that give $s = 2$ are presented. For both systems, the main filter h satisfying

$$h(\pm 1) = \frac{1}{2}, \quad h(m) = 0, \quad \text{for } m \neq \pm 1,$$

is used. L-CAMP systems with higher performance can be found in [4].

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